Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Generalized Exponential Distribution

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the exponentiated generalized exponential distribution. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distribution. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distribution. The method can be extended to other probability distributions and can serve an alternative to approximation.**

***Index Terms*— Exponentiated, exponential distribution, reversed hazard function, calculus, differentiation.**

I. Introduction

THIS distribution was proposed by Oguntunde et al. [1] as a three parameter model that can be used as one of the generalizations of the exponential distribution. The proposed model has also the generalized exponential and exponentiated exponential distribution as its submodels. The distribution belongs to the exponentiated class of distributions which has seen a lot of research activities. Details on the general class of exponentiated distributions can be seen in [2-6].

In particular, some exponentiated distributions are available in scientific literature such as: exponentiated Gumbel type-2 distribution [7], exponentiated Weibull distribution [8-10], exponentiated generalized inverted exponential distribution [11], exponentiated generalized inverse Gaussian distribution [12], exponentiated generalized inverse Weibull distribution [13-14], gamma-exponentiated exponential distribution [15], exponentiated Gompertz distribution [16-17], beta Exponentiated Mukherjii-Islam Distribution [18], transmuted exponentiated Pareto-i distribution [19], gamma exponentiated exponential–Weibull distribution [20], exponentiated gamma distribution [21], exponentiated Gumbel distribution [22], exponentiated uniform distribution [23] and beta exponentiated Weibull distribution [24-25]. Others are: exponentiated log-logistic distribution [26], McDonald exponentiated gamma distribution [27], exponentiated Generalized Weibull Distribution [28], beta exponentiated gamma distribution [29], exponentiated gamma distribution [30], exponentiated Pareto distribution [31], exponentiated Kumaraswamy distribution [32], exponentiated modified Weibull extension distribution [33], exponentiated Weibull-Pareto distribution [34], exponentiated lognormal distribution [35], exponentiated Perks distribution [36] and Kumaraswamy-transmuted exponentiated modified Weibull distribution [37]. Also available are: exponentiated power Lindley–Poisson distribution [38], exponentiated Chen distribution [39], exponentiated reduced Kies distribution [40], exponentiated inverse Weibull geometric distribution [41], exponentiated geometric distribution [42-43], exponentiated Weibull geometric distribution [44], exponentiated transmuted Weibull geometric distribution [45], exponentiated half logistic distribution [46], transmuted exponentiated Gumbel distribution [47], exponentiated Kumaraswamy-power function distribution [48], exponentiated-log-logistic geometric distribution [49], bivariate exponentaited generalized Weibull-Gompertz distribution [50] and so on.

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of exponentiated generalized exponential distribution by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [51], beta distribution [52], raised cosine distribution [53], Lomax

Manuscript received June 30, 2017; revised July 25, 2017. This work was sponsored by Covenant University, Ota, Nigeria.

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distribution [54], beta prime distribution or inverted beta distribution [55].

II. Probability Density Function

The probability density function of the exponentiated generalized exponential distribution is given as;

$$f(x) = \alpha\beta\,\mathrm{e}^{-\alpha\lambda x}[(1-\mathrm{e}^{-\lambda x})^{\alpha}]^{\beta-1}$$

(1) To obtain the first order ordinary differential equation for the probability density function of the exponentiated generalized exponential distribution, differentiate equation (1), to obtain;

$$f'(x) = \left\{ \begin{array}{l} \dfrac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-1}[(1-\mathrm{e}^{-\lambda x})^{\alpha}]^{\beta-2}}{[(1-\mathrm{e}^{-\lambda x})^{\alpha}]^{\beta-1}} \\ -\dfrac{\alpha\lambda\,\mathrm{e}^{-\alpha\lambda x}}{\mathrm{e}^{-\alpha\lambda x}} \end{array} \right\} f(x) \tag{2}$$

The condition necessary for the existence of the equation is $x, \alpha, \beta, \lambda > 0$.

$$f'(x) = \left\{ -\alpha\lambda + \frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-1}}{(1-\mathrm{e}^{-\lambda x})^{\alpha}} \right\} f(x) \tag{3}$$

$$f'(x) = \left\{ -\alpha\lambda + \frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})} \right\} f(x) \tag{4}$$

Differentiate equation (4) to obtain;

$$f''(x) = \left\{ -\alpha\lambda + \frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})} \right\} f'(x) - \left\{ \frac{\alpha(\beta-1)\lambda^2(\mathrm{e}^{-\lambda x})^2}{(1-\mathrm{e}^{-\lambda x})^2} + \frac{\alpha(\beta-1)\lambda^2\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})} \right\} f(x) \tag{5}$$

The condition necessary for the existence of the equation is $x, \alpha, \beta, \lambda > 0$.

The following equations obtained from equation (4) are needed to simplify equation (5);

$$-\alpha\lambda + \frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})} = \frac{f'(x)}{f(x)} \tag{6}$$

$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})} = \frac{f'(x)}{f(x)} + \alpha\lambda \tag{7}$$

$$\left(\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})} \right)^2 = \left(\frac{f'(x)}{f(x)} + \alpha\lambda \right)^2 \tag{8}$$

$$\frac{\alpha(\beta-1)(\lambda\,\mathrm{e}^{-\lambda x})^2}{(1-\mathrm{e}^{-\lambda x})^2} = \frac{1}{\alpha(\beta-1)} \left(\frac{f'(x)}{f(x)} + \alpha\lambda \right)^2 \tag{9}$$

$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})} = \lambda \left(\frac{f'(x)}{f(x)} + \alpha\lambda \right) \tag{10}$$

Substitute equations (6), (9) and (10) into equation (5);

$$f''(x) = \frac{f'^2(x)}{f(x)} - \left\{ \begin{array}{l} \dfrac{1}{\alpha(\beta-1)} \left(\dfrac{f'(x)}{f(x)} + \alpha\lambda \right)^2 \\ +\lambda \left(\dfrac{f'(x)}{f(x)} + \alpha\lambda \right) \end{array} \right\} f(x) \tag{11}$$

The condition necessary for the existence of the equation is $x, \alpha, \lambda > 0, \beta > 1$.

The ordinary differential equations can be obtained for the particular values of the parameters.

III. Quantile Function

The Quantile function of the exponentiated generalized exponential distribution is given as;

$$Q(p) = \frac{1}{\alpha\lambda} \ln \left(\frac{1}{1 - p^{\frac{1}{\beta}}} \right) \tag{12}$$

To obtain the first order ordinary differential equation for the Quantile function of the exponentiated generalized exponential distribution, differentiate equation (12), to obtain;

$$Q'(p) = \frac{p^{\frac{1}{\beta}-1}}{\alpha\beta\lambda(1 - p^{\frac{1}{\beta}})} \tag{13}$$

The condition necessary for the existence of the equation is $\alpha, \beta, \lambda > 0, 0 < p < 1$.

$$\alpha\beta\lambda(1 - p^{\frac{1}{\beta}})Q'(p) - p^{\frac{1}{\beta}-1} = 0 \tag{14}$$

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered in shown in **Table 1.**

Table 1: Classes of differential equations obtained for the quantile function of exponentiated generalized exponential distribution for different parameters.

α	β	λ	Ordinary differential equation
1	1	1	$(1-p)Q'(p)-1=0$
1	1	2	$2(1-p)Q'(p)-1=0$
1	2	1	$2(1-p)Q'(p)-1=0$
1	2	2	$4(1-p)Q'(p)-1=0$
2	1	1	$2(\sqrt{p}-p)Q'(p)-1=0$
2	1	2	$4(\sqrt{p}-p)Q'(p)-1=0$
2	2	1	$4(\sqrt{p}-p)Q'(p)-1=0$
2	2	2	$8(\sqrt{p}-p)Q'(p)-1=0$

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

IV. SURVIVAL FUNCTION

The survival function of the exponentiated generalized exponential distribution is given as;

$$S(t)=1-[1-\mathrm{e}^{-\alpha\lambda x}]^{\beta} \quad (15)$$

To obtain the first order ordinary differential equation for the survival function of the exponentiated generalized exponential distribution, differentiate equation (15), to obtain;

$$S'(t)=-\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda x}[1-\mathrm{e}^{-\alpha\lambda x}]^{\beta-1} \quad (16)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0$.

Equation (16) can be written as;

$$[1-\mathrm{e}^{-\alpha\lambda x}]^{\beta}=1-S(t) \quad (17)$$

Substitute equation (17) into equation (16) to obtain;

$$S'(t)=-\frac{\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda x}(1-S(t))}{[1-\mathrm{e}^{-\alpha\lambda x}]}$$

(18) From equation (17),

$$1-\mathrm{e}^{-\alpha\lambda x}=(1-S(t))^{\frac{1}{\beta}} \quad (19)$$

$$\mathrm{e}^{-\alpha\lambda x}=1-(1-S(t))^{\frac{1}{\beta}} \quad (20)$$

Substitute equations (19) and (20) into equation (18);

$$S'(t)=-\frac{\alpha\beta\lambda(1-(1-S(t))^{\frac{1}{\beta}})(1-S(t))}{(1-S(t))^{\frac{1}{\beta}}} \quad (21)$$

$$S'(t)=-\alpha\beta\lambda(1-(1-S(t))^{\frac{1}{\beta}})(1-S(t))^{1-\frac{1}{\beta}} \quad (22)$$

$$S'(t)+\alpha\beta\lambda(1-(1-S(t))^{\frac{1}{\beta}})(1-S(t))^{1-\frac{1}{\beta}}=0 \quad (23)$$

$$S(1)=1-[1-\mathrm{e}^{-\alpha\lambda}]^{\beta} \quad (24)$$

The ordinary differential equations can be obtained for the different values of the parameters.

When $\beta=1$, equations (23) and (24) become;

$$S'(t)+\alpha\lambda S(t)=0 \quad (25)$$

$$S(1)=\mathrm{e}^{-\alpha\lambda} \quad (26)$$

When $\beta=2$, equations (23) and (24) become;

$$(\sqrt{1-S(t)})S'(t)+2\alpha\lambda(1-\sqrt{1-S(t)})=0 \quad (27)$$

$$S(1)=1-[1-\mathrm{e}^{-\alpha\lambda}]^{2} \quad (28)$$

V. INVERSE SURVIVAL FUNCTION

The inverse survival function of the exponentiated generalized exponential distribution is given as;

$$Q(p)=\frac{1}{\alpha\lambda}\ln\left(\frac{1}{1-(1-p)^{\frac{1}{\beta}}}\right)$$

(29) To obtain the first order ordinary differential equation for the inverse survival function of the exponentiated generalized exponential distribution, differentiate equation (29), to obtain;

$$Q'(p)=-\frac{p^{\frac{1}{\beta}-1}}{\alpha\beta\lambda(1-(1-p)^{\frac{1}{\beta}})} \quad (30)$$

The condition necessary for the existence of the equation is $\alpha,\beta,\lambda>0, 0<p<1$.

$$\alpha\beta\lambda(1-(1-p)^{\frac{1}{\beta}})Q'(p)+p^{\frac{1}{\beta}-1}=0 \quad (31)$$

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered and shown in **Table 2**.

Table 2: Classes of differential equations obtained for the inverse survival function of exponentiated generalized exponential distribution for different parameters

α	β	λ	Ordinary differential equation
1	1	1	$pQ'(p)+1=0$
1	1	2	$2pQ'(p)+1=0$
1	2	1	$2pQ'(p)+1=0$
1	2	2	$4pQ'(p)+1=0$
2	1	1	$2\sqrt{p}(1-\sqrt{1-p})Q'(p)+1=0$
2	1	2	$4\sqrt{p}(1-\sqrt{1-p})Q'(p)+1=0$
2	2	1	$4\sqrt{p}(1-\sqrt{1-p})Q'(p)+1=0$
2	2	2	$8\sqrt{p}(1-\sqrt{1-p})Q'(p)+1=0$

The complexity of the ODE increases as the value of the parameters changes.

VI. HAZARD FUNCTION

The hazard function of the exponentiated generalized exponential distribution is given as;

$$h(t)=\frac{\alpha\beta\,\mathrm{e}^{-\alpha\lambda t}[(1-\mathrm{e}^{-\lambda t})^{\alpha}]^{\beta-1}}{1-[1-\mathrm{e}^{-\alpha\lambda t}]^{\beta}} \quad (32)$$

To obtain the first order ordinary differential equation for the hazard function of the exponentiated generalized exponential distribution, differentiate equation (32), to obtain;

$$h'(t)=-\frac{\alpha\lambda\,\mathrm{e}^{-\alpha\lambda t}}{\mathrm{e}^{-\alpha\lambda t}}h(t)+$$
$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}(1-\mathrm{e}^{-\lambda t})^{\alpha-1}[(1-\mathrm{e}^{-\lambda t})^{\alpha}]^{\beta-2}}{[(1-\mathrm{e}^{-\lambda t})^{\alpha}]^{\beta-1}}h(t) \quad (33)$$
$$+\frac{\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda t}(1-\mathrm{e}^{-\alpha\lambda t})^{\beta-1}[1-(1-\mathrm{e}^{-\alpha\lambda t})^{\beta}]^{-2}}{[1-(1-\mathrm{e}^{-\alpha\lambda t})^{\beta}]^{-1}}h(t)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0$.

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$h'(t)=\left\{\begin{array}{l}\dfrac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}(1-\mathrm{e}^{-\lambda t})^{\alpha-1}}{(1-\mathrm{e}^{-\lambda t})^{\alpha}}\\+\dfrac{\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda t}(1-\mathrm{e}^{-\alpha\lambda t})^{\beta-1}}{[1-(1-\mathrm{e}^{-\alpha\lambda t})^{\beta}]}-\alpha\lambda\end{array}\right\}h(t) \quad (34)$$

$$h'(t)=\left\{-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}+h(t)\right\}h(t) \quad (35)$$

Differentiate equation (35) to obtain;

$$h''(t)=\left\{\begin{array}{l}h'(t)-\dfrac{\alpha(\beta-1)\lambda^{2}(\mathrm{e}^{-\lambda t})^{2}}{(1-\mathrm{e}^{-\lambda t})^{2}}\\+\dfrac{\alpha(\beta-1)\lambda^{2}\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}\end{array}\right\}h(t)+\left\{-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}+h(t)\right\}h'(t) \quad (36)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0.$

The following equations obtained from equation (35) are needed to simplify equation (36);

$$-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}+h(t)=\frac{h'(t)}{h(t)} \quad (37)$$

$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}=\frac{h'(t)}{h(t)}+\alpha\lambda-h(t) \quad (38)$$

$$\left(\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}\right)^{2}=\left(\frac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right)^{2} \quad (39)$$

$$\frac{\alpha(\beta-1)(\lambda\,\mathrm{e}^{-\lambda t})^{2}}{(1-\mathrm{e}^{-\lambda t})^{2}}=\frac{1}{\alpha(\beta-1)}\left(\frac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right)^{2} \quad (40)$$

$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}=\lambda\left(\frac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right) \quad (41)$$

Substitute equations (37), (40) and (41) into equation (36);

$$h''(t)=\frac{h'^{2}(t)}{h(t)}-\left\{\begin{array}{l}\dfrac{1}{\alpha(\beta-1)}\left(\dfrac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right)^{2}\\+\lambda\left(\dfrac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right)-h(t)\end{array}\right\}h(t) \quad (42)$$

The condition necessary for the existence of the equation is $t,\alpha,\lambda>0,\beta>1.$

The ordinary differential equations can be obtained for the particular values of the parameters.

VII. Reversed Hazard Function

The reversed hazard function of the exponentiated generalized exponential distribution is given as;

$$j(t)=\frac{\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda t}}{[1-\mathrm{e}^{-\lambda t}]^{\alpha}}$$

(43) To obtain the first order ordinary differential equation for the reversed hazard function of the exponentiated generalized exponential distribution, differentiate equation (43), to obtain;

$$j'(t)=\left\{-\frac{\alpha\lambda\,\mathrm{e}^{-\alpha\lambda t}}{\mathrm{e}^{-\alpha\lambda t}}-\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}[1-\mathrm{e}^{-\lambda t}]^{-(\alpha+1)}}{[1-\mathrm{e}^{-\lambda t}]^{-\alpha}}\right\}j(t) \quad (44)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0.$

$$j'(t)=-\left\{\alpha\lambda+\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right\}j(t) \quad (45)$$

Differentiate equation (45);

$$j''(t)=\left\{-\alpha\lambda-\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right\}j'(t)+\left\{\frac{\alpha(\lambda\,\mathrm{e}^{-\lambda t})^{2}}{(1-\mathrm{e}^{-\lambda t})^{2}}+\frac{\alpha\lambda^{2}\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right\}j(t) \quad (46)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0.$

The following equations obtained from equation (45) are needed to simplify equation (46);

$$\frac{j'(t)}{j(t)}=-\alpha\lambda-\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}} \quad (47)$$

$$\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}=-\frac{j'(t)}{j(t)}-\alpha\lambda \quad (48)$$

$$\left(\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right)^{2}=\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right)^{2} \quad (49)$$

$$\frac{\alpha(\lambda\,\mathrm{e}^{-\lambda t})^{2}}{(1-\mathrm{e}^{-\lambda t})^{2}}=\frac{1}{\alpha}\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right)^{2} \quad (50)$$

$$\frac{\alpha\lambda^{2}\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}=-\lambda\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right) \quad (51)$$

Substitute equations (47), (50) and (51) into equation (46);

$$j''(t)=\frac{j'^{2}(t)}{j(t)}+\left\{\frac{1}{\alpha}\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right)^{2}-\lambda\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right)\right\}j(t) \quad (52)$$

The ordinary differential equations can be obtained for the particular values of the parameters.

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [56-69], especially for the cases of the quantile and inverse survival functions. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

VIII. Concluding Remarks

In this paper, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the exponentiated generalized exponential distribution. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs. Furthermore, the complexity of the ODEs depends greatly on the values of the parameters.

Acknowledgment

The authors are unanimous in appreciation of financial sponsorship from Covenant University. The constructive suggestions of the reviewers are greatly appreciated.

References

[1] P.E. Oguntunde, A.O. Adejumo and K.A. Adepoju, "Assessing the flexibility of the exponentiated generalized exponential distribution", *Pacific J. Sci. Tech.*, vol. 17, no. 1, pp. 49-57, 2016.
[2] G.M. Cordeiro, A.Z. Afify, H.M. Yousof, R.R. Pescim and G.R. Aryal, "The exponentiated Weibull-H family of distributions: Theory and applications", *Medit. J. Math.*, vol. 14, no. 4, 155, 2017.
[3] S. Nadarajah and S. Kotz, "The exponentiated type distributions", *Acta Applic. Math.*, vol. 92, no. 2, pp. 97-111, 2006.
[4] G.M. Cordeiro, E.M. Ortega and D.C.C. da Cuncha, "The exponentiated generalized class of distributions", *J. Data Sci.*, vol. 11, pp. 1-27, 2013.
[5] S. Rezaei, A.K. Marvasty, S. Nadarajah and M. Alizadeh, "A new exponentiated class of distributions: Properties and applications", *Comm. Stat. Theo. Meth.*, vol. 46, no. 12, pp. 6054-6073, 2017.
[6] R.D. Gupta and D. Kundu, "Exponentiated exponential family: an alternative to gamma and Weibull distributions", *Biometrical J.*, vol. 43, no. 1, pp. 117-130, 2001.
[7] I.E. Okorie, IA.C. Akpanta and J. Ohakwe, "The exponentiated Gumbel Type-2 distribution: properties and application", *Int. J. Math. Math. Sci.*, Art. nọ. 5898356, 2016.
[8] M. Pal, M.M. Ali and J. Woo, "Exponentiated Weibull distribution", *Statistica*, vol. 66, no. 2, pp. 139-147, 2006.
[9] G.S. Mudholkar and D.K. Srivastava, "Exponentiated Weibull family for analyzing bathtub failure-rate data", *IEEE Trans. Relia.*, vol. 42, no. 2, pp. 299-302, 1993.
[10] M.N. Nassar and F.H. Eissa, "On the exponentiated Weibull distribution", *Comm. Stat. Theo. Meth.*, vol. 32, no. 7, pp. 1317-1336, 2003.
[11] P.E. Oguntunde, A.O. Adejumo and O.S. Balogun, "Statistical properties of the exponentiated generalized inverted exponential distribution", *Appl. Math.*, vol. 4, no. 2, pp. 47-55, 2014.
[12] A.J. Lemonte and G.M. Cordeiro, "The exponentiated generalized inverse Gaussian distribution", *Stat. Prob. Lett.*, vol. 81, no. 4, pp. 506-517, 2011.
[13] A. Flaih, H. Elsalloukh, E. Mendi and M. Milanova, "The exponentiated inverted Weibull distribution", *Appl. Math. Inf. Sci*, vol. 6, no. 2, pp. 167-171, 2012.
[14] I. Elbatal and H.Z. Muhammed, "Exponentiated generalized inverse Weibull distribution", *Appl. Math. Sci.*, vol. 8, no. 81, pp. 3997-4012, 2014.
[15] M.M. Ristić and N. Balakrishnan, "The gamma-exponentiated exponential distribution", *J. Stat. Comput. Simul.*, vol. 82, no. 8, pp. 1191-1206, 2012.
[16] H.H. Abu-Zinadah and A.S. Aloufi, "Some characterizations of the exponentiated Gompertz distribution", *Int. Math. Forum*, vol. 9, no. 30, pp. 1427-1439, 2014.
[17] H.H. Abu-Zinadah, "Six method of estimations for the shape parameter of exponentiated Gompertz distribution", *Appl. Math. Sci.*, vol. 8, no. 85-88, pp. 4349-4359, 2014.
[18] S.A. Siddiqui, S. Dwivedi, P. Dwivedi and M. Alam, "Beta exponentiated Mukherjii-Islam distribution: Mathematical study of different properties", *Global J. Pure Appl. Math.*, vol. 12, no. 1, pp. 951-964, 2016.
[19] A. Fatima and A. Roohi, "Transmuted exponentiated Pareto-i distribution", *Pak. J. Statist*, vol. 32, no. 1, pp. 63-80, 2015.
[20] T.K. Pogány and A. Saboor, "The Gamma exponentiated exponential–Weibull distribution", *Filomat*, vol. 30, no. 12, pp. 3159-3170, 2016.
[21] S. Nadarajah and A.K. Gupta, "The exponentiated gamma distribution with application to drought data", *Calcutta Stat. Assoc. Bul.*, vol. 59, no. 1-2, pp. 29-54, 2007.
[22] S. Nadarajah, "The exponentiated Gumbel distribution with climate application", *Environmetrics*, vol. 17, no. 1, pp. 13-23, 2006.
[23] C.S. Lee and H.Y. Won, "Inference on reliability in an exponentiated uniform distribution", *J. Korean Data Info. Sci. Soc.*, vol. 17, no. 2, pp. 507-513, 2006.
[24] G.M. Cordeiro, A.E. Gomes, C.Q. da-Silva and E.M. Ortega, "The beta exponentiated Weibull distribution", *J. Stat. Comput. Simul.*, vol. 83, no. 1, pp. 114-138, 2013.
[25] S. Hashmi and A.Z. Memon, "Beta exponentiated Weibull distribution (its shape and other salient characteristics)", *Pak. J. Stat.*, vol. 32, no. 4, pp. 301-327, 2016.
[26] K. Rosaiah, R.R.L. Kantam and S. Kumar, "Reliability test plans for exponentiated log-logistic distribution", *Econ. Qual. Control*, vol. 21, no. 2, pp. 279-289, 2006.
[27] A.A. Al-Babtain, F. Merovci and I. Elbatal, "The McDonald exponentiated gamma distribution and its statistical properties", SpringerPlus, vol. 4, no. 1, art. 2, 2015.
[28] P.E. Oguntunde, O.A. Odetunmibi and A.O. Adejumo, "On the Exponentiated Generalized Weibull Distribution: A Generalization of the Weibull Distribution", *Indian J. Sci. Tech.*, vol. *8*, no. 35, 2015.
[29] N. Feroze and I. Elbatal, "Beta exponentiated gamma distribution: some properties and estimation", *Pak. J. Stat. Oper. Res.*, vol. 12, no. 1, pp. 141-154, 2016.
[30] A.I. Shawky and R.A. Bakoban, Exponentiated gamma distribution: Different methods of estimations, *J. Appl. Math.*, vol. 2012, art. no. 284296, 2012.
[31] A.I. Shawky and H.H. Abu-Zinadah, Exponentiated Pareto distribution: Different method of estimations, *Int. J. Contem. Math. Sci.*, vol. 4, no. 14, pp. 677- 693, 2009.
[32] A.J. Lemonte, W. Barreto-Souza and G.M. Cordeiro, The exponentiated Kumaraswamy distribution and its log-transform, *Braz. J. Prob. Stat.*, vol. 27, no. 1, pp. 31-53, 2013.
[33] A.M. Sarhan and J. Apaloo, "Exponentiated modified Weibull extension distribution", *Relia. Engine. Syst. Safety*, vol. 112, pp. 137-144, 2013.
[34] A.Z. Afify, H.M. Yousof, G.G. Hamedani and G. Aryal, "The exponentiated Weibull-Pareto distribution with application", *J. Stat. Theory Appl.*, vol. 15, pp. 328-346, 2016.
[35] C.S. Kakde and D.T. Shirke, "On exponentiated lognormal distribution", *Int. J. Agric. Stat. Sci.*, vol. 2, pp. 319-326, 2006.
[36] B. Singh and N. Choudhary, "The exponentiated Perks distribution", *Int. J. Syst. Assur. Engine. Magt.*, vol. 8, no. 2, pp. 468-478, 2017.
[37] A. Al-Babtain, A.A. Fattah, A.H.N. Ahmed and F. Merovci, "The Kumaraswamy-transmuted exponentiated modified Weibull distribution", *Comm. Stat. Simul. Comput.*, vol. 46, no. 5, pp. 3812-3832, 2017.
[38] M. Pararai, G. Warahena-Liyanage and B.O. Oluyede, "Exponentiated power Lindley–Poisson distribution: Properties and applications", *Comm. Stat. Theo. Meth.*, vol. 46, no. 10, pp. 4726-4755, 2017.
[39] S. Dey, D. Kumar, P.L. Ramos and F. Louzada, "Exponentiated Chen distribution: Properties and estimation", *Comm. Stat. Simul. Comput.*, To appear, 2017.
[40] C.S. Kumar and S.H.S. Dharmaja, "The exponentiated reduced Kies distribution: Properties and applications", *Comm. Stat. Theo. Meth.*, To appear, 2017.
[41] Y. Chung, D.K. Dey and M. Jung, "The exponentiated inverse Weibull geometric distribution", *Pak. J. Stat.*, vol. 33, no. 3, pp. 161-178.
[42] S. Nadarajah and S.A.A. Bakar, "An exponentiated geometric distribution", *Appl. Math. Model.*, vol. 40, no. 13, pp. 6775-6784, 2016.
[43] V. Nekoukhou, M.H. Alamatsaz and H. Bidram, "A note on exponentiated geometric distribution: Another generalization of geometric distribution", *Comm. Stat. Theo. Meth.*, vol. 45, no. 5, pp. 1575-1575, 2016.
[44] Y. Chung and Y. Kang, "The exponentiated Weibull geometric distribution: Properties and Estimations", *Comm. Stat. Appl. Meth.*, vol. 21, no. 2, pp. 147-160, 2014.

[45] A.A. Fattah, S. Nadarajah and A.H.N. Ahmed, "The exponentiated transmuted Weibull geometric distribution with application in survival analysis", *Comm. Stat. Simul. Comput.*, To appear, 2017.

[46] W. Gui, "Exponentiated half logistic distribution: different estimation methods and joint confidence regions", *Comm. Stat. Simul. Comput.*, To appear, 2017.

[47] D. Deka, B. Das and B.K. Baruah, B. "Transmuted exponentiated Gumbel distribution (TEGD) and its application to water quality data", *Pak. J. Stat. Oper. Res.*, vol. 13, no. 1, pp. 115-126, 2017.

[48] N. Bursa and G. Ozel, "The exponentiated Kumaraswamy-power function distribution", *Hacettepe J. Math. Stat.*, vol. 46, no. 2, pp. 277-292, 2017.

[49] N.V. Mendoza, E.M. Ortega and G.M. Cordeiro, "The exponentiated-log-logistic geometric distribution: Dual activation", *Comm. Stat. Theo. Meth.*, vol. 45, no. 13, pp. 3838-3859, 2016.

[50] A.H. El-Bassiouny, M.A., EL-Damcese, A. Mustafa and M.S. Eliwa, "Bivariate exponentaited generalized Weibull-Gompertz distribution", *J. Appl. Prob.*, vol. 11, no. 1, pp. 25-46, 2016.

[51] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous univariate distributions, Wiley New York. ISBN: 0-471-58495-9, 1994.

[52] W.P. Elderton, Frequency curves and correlation, Charles and Edwin Layton. London, 1906.

[53] H. Rinne, Location scale distributions, linear estimation and probability plotting using MATLAB, 2010.

[54] N. Balakrishnan and C.D. Lai, Continuous bivariate distributions, 2nd edition, Springer New York, London, 2009.

[55] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Volume 2. 2nd edition, Wiley, 1995.

[56] S.O. Edeki, H.I. Okagbue , A.A. Opanuga and S.A. Adeosun, "A semi - analytical method for solutions of a certain class of second order ordinary differential equations", *Applied Mathematics*, vol. 5, no. 13, pp. 2034 – 2041, 2014.

[57] S.O. Edeki, A.A Opanuga and H.I Okagbue, "On iterative techniques for numerical solutions of linear and nonlinear differential equations", *J. Math. Computational Sci*., vol. 4, no. 4, pp. 716-727, 2014.

[58] A.A. Opanuga, S.O. Edeki, H.I. Okagbue, G.O. Akinlabi, A.S. Osheku and B. Ajayi, "On numerical solutions of systems of ordinary differential equations by numerical-analytical method", *Appl. Math. Sciences*, vol. 8, no. 164, pp. 8199 – 8207, 2014.

[59] S.O. Edeki , A.A. Opanuga, H.I. Okagbue , G.O. Akinlabi, S.A. Adeosun and A.S. Osheku, "A Numerical-computational technique for solving transformed Cauchy-Euler equidimensional equations of homogenous type. *Adv. Studies Theo. Physics*, vol. 9, no. 2, pp. 85 – 92, 2015.

[60] S.O. Edeki , E.A. Owoloko , A.S. Osheku , A.A. Opanuga , H.I. Okagbue and G.O. Akinlabi, "Numerical solutions of nonlinear biochemical model using a hybrid numerical-analytical technique", *Int. J. Math. Analysis*, vol. 9, no. 8, pp. 403-416, 2015.

[61] A.A. Opanuga , S.O. Edeki , H.I. Okagbue and G.O. Akinlabi, "Numerical solution of two-point boundary value problems via differential transform method", *Global J. Pure Appl. Math.*, vol. 11, no. 2, pp. 801-806, 2015.

[62] A.A. Opanuga, S.O. Edeki, H.I. Okagbue and G. O. Akinlabi, "A novel approach for solving quadratic Riccati differential equations", *Int. J. Appl. Engine. Res.*, vol. 10, no. 11, pp. 29121-29126, 2015.

[63] A.A Opanuga, O.O. Agboola and H.I. Okagbue, "Approximate solution of multipoint boundary value problems", *J. Engine. Appl. Sci.*, vol. 10, no. 4, pp. 85-89, 2015.

[64] A.A. Opanuga, O.O. Agboola, H.I. Okagbue and J.G. Oghonyon, "Solution of differential equations by three semi-analytical techniques", *Int. J. Appl. Engine. Res.*, vol. 10, no. 18, pp. 39168-39174, 2015.

[65] A.A. Opanuga, H.I. Okagbue, S.O. Edeki and O.O. Agboola, "Differential transform technique for higher order boundary value problems", *Modern Appl. Sci.*, vol. 9, no. 13, pp. 224-230, 2015.

[66] A.A. Opanuga, S.O. Edeki, H.I. Okagbue, S.A. Adeosun and M.E. Adeosun, "Some Methods of Numerical Solutions of Singular System of Transistor Circuits", J. *Comp. Theo. Nanosci.*, vol. 12, no. 10, pp. 3285-3289, 2015.

[67] A.A. Opanuga, E.A. Owoloko and H.I. Okagbue, "Comparison homotopy perturbation and Adomian decomposition techniques for parabolic equations" *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017,* 5-7 July, 2017, London, U.K., pp24-27

[68] A.A. Opanuga, E.A. Owoloko, H.I. Okagbue and O.O. Agboola, "Finite difference method and Laplace transform for boundary value problems", *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp65-69*

[69] A.A. Opanuga, E.A. Owoloko, O.O. Agboola and H.I. Okagbue, "Application of homotopy perturbation and modified Adomian decomposition methods for higher order boundary value problems", *Lecture Notes in Engineering and Computer Science: Proceedings of The World Congress on Engineering 2017, 5-7 July, 2017, London, U.K., pp130-134*